A NEW PROOF OF THE STRONG GOLDBACH CONJECTURE

by

PAUL S. BRUCKMAN
38 Front Street, Unit #302
Nanaimo, BC V9R 0B8 (Canada)
Phone : (250) 591-2290
e-mail : pbruckman@hotmail.com

ABSTRACT :

An elementary proof of the strong version of Goldbach's Conjecture (SGC) is presented. The Golbach Sets of order k, certain nested sets $\{S_k\}$ of natural numbers, are defined, with minimal elements R_k. It is shown that the smallest counterexample R to SGC, if any, must be an element of the sequence $\{R_k\}$, say R_M, where $R_M < p_M < 2R_M < p_{M+1}$, and p_k is the k-th odd prime. By introducing certain auxiliary sequences, it is shown that $R_k > e^{\sqrt{k}}$, hence $R_k > p_k$ for all $k \geq 1$. This last result contradicts the result previously deduced for the putative value R_M, thereby establishing SGC .

Key Words or Terms :

prime, characteristic function, counting function, Goldbach Function, Goldbach Set of order k, normal, abnormal, Strong Goldbach Conjecture, Extended Goldbach Conjecture.

A NEW PROOF OF THE STRONG GOLDBACH CONJECTURE

by

Paul S. Bruckman
38 Front Street, Unit #302, BC V9R 0B8 (Canada)

Phone: (250) 591-2290
e-mail : pbruckman@hotmail.com

1. INTRODUCTION :

In [1], this author made some sporadic efforts, intended to indicate a possible method for proving the Strong Goldbach Conjecture (SGC). These efforts have evolved into a more complete theory, reported in this article, along with a proof of this famous conjecture, which has at last been found. Also, this paper amends and supersedes a previous paper by the author [2], due to an error in that previous paper that was only discovered after its publication.

As indicated in [1] and [2], we may state SGC as follows :

STRONG GOLDBACH CONJECTURE (SGC) :

Every even integer $n \geq 6$ is the sum of two odd (not necessarily distinct) primes.

A good account of the history of this problem, along with a summary of the relevant research associated with it, may be found in the introduction of W. Yuan's paper [6]. The approach given in this paper is elementary, and avoids use of the "circle" methods employed by some previous authors.

First, we use the notation $\delta(n)$ to denote the characteristic function of the odd positive primes, thus adopting the convention that $\delta(n) = 0$ if $n \leq 2$. The counting

function of the primes, normally denoted by $\pi(x)$, is then given by :

$$\pi(x) = 1 + \sum_{3 \leq n \leq x} \delta(n) \qquad (1)$$

Here, x is any positive real number, generally thought of as large. Also, let $\pi^*(x) = \pi(x) - 1$. Throughout this paper, the lower-case letters p (usually indexed) and q denote odd primes ; other letters denote positive integers, unless otherwise indicated (thus, x and C_k denote continuous and positive variables).

We let $\theta(2n)$ represent the number of ways of writing 2n as the sum of two (not necessarily distinct) odd primes p and q, where p+q = q+p = 2n are two distinct ways, unless p = q . For example, $\theta(6) = 1$, since 6 = 3+3 uniquely. However, $\theta(8) = 2$, since 8 = 3+5 = 5+3 . Also, $\theta(10) = 3$, since 10 = 3+7 = 5+5 = 7+3, etc. Note that $\theta(2n)$ is odd iff n is prime, in which case $\theta(2n) \geq 1$. We see that $\theta(2n) = \sum_{k=3}^{2n-3} \delta(k)\, \delta(2n-k)$. For brevity, we call $\theta(2n)$ the *Goldbach Function* of 2n .

Given this introduction, SGC may be rephrased as follows :

STRONG GOLDBACH CONJECTURE :

$$\theta(2n) \geq 1 \text{ for all } n \geq 3$$

2. THE PRIMARY HYPOTHESIS :

We postulate the existence of a positive integer R such that $\theta(2R) = 0$. Moreover, let us assume that R is the smallest such integer . We refer to this hypothesis as the *primary hypothesis* (PH) . We will proceed from PH and eventually establish a *reductio ad absurdum*, which will thereby prove SGC. We number the odd primes sequentially as follows : $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, etc. Next, we introduce the following definition of the *Goldbach Set of order k* :

<u>DEFINITION 1</u> : Given an integer $k \geq 1$, let $\mathfrak{S}_k$ denote the set of integers $n \geq 6$ such that the following two conditions are met :

(i) $2n > p_k$; and

(ii) $\delta(2n - p_i) = 0$, $1 \leq i \leq k$

$\mathfrak{S}_k$ thus defined is called the *Goldbach Set of order k*.

Note that this definition ensures that $2n - p_i$ is composite (or possibly = 1), for i = 1, 2, …, k . We also see, applying the definition, that $\mathfrak{S}_1 \supseteq \mathfrak{S}_2 \supseteq \mathfrak{S}_3 \supseteq \ldots$. However, at this point, it is not clear that such sets are well-defined, or even that they exist ; hence, our first task is to prove that DEFINITION 1 is not vacuous.

Define a sequence $\{P_k\}$ as follows : $P_1 = 6$, $P_k = 3 \cdot 5 \cdot 7 \cdot \ldots \cdot p_k$, k = 2, 3, …. We note that $\delta(2P_k - p_i) = 0$, $1 \leq i \leq k$. It is also clear that $2P_k > p_k$, k = 1, 2, … , and that $\{P_k\}$ is strictly increasing. Therefore, $P_k \in \mathfrak{S}_k$, showing that the sets $\mathfrak{S}_k$ are indeed well-defined and do exist . Moreover, since $P_k \in \mathfrak{S}_k$, $2P_k \in \mathfrak{S}_k$, $3P_k \in \mathfrak{S}_k$, etc., the sets $\mathfrak{S}_k$ are infinite.

By way of illustration, $\mathfrak{S}_1$ = {6, 9, 12, 14, 15, 18, 19, 21, 24, 26, 27, 29, 30, 33, 34, 36, 39, 40, 42, 44, 45, 47, 48, 49, 51, 54, 57, 59, 60, 61, 62, 63, 64, 66, 68, 69, 72, 73, 74, 75, 78, 79, 81, 82, 84, 86, 87, 89, 90, 93, 94, 95, 96, 99, 102, 103, 104, 105, 106, 108, 109, 110, 111, 112, 114, 117, 119, 120, 123, 124, 125, 126, 128, 129, 131, 132, 134, 135, 138, 139, 141, 144, 145, 146, 147, 149, 150, 151, 152, 153, 154, 156, 159, 161, 162, 163, 164, 165, 166, 168, 169, 171, 172, 173, 174, 177, 179, 180, 182, 183, 184, 186, 187, 189, 190, 192, 194, 195, 197, 198, 199, 201, …} ;

$\mathfrak{S}_2$ = {15, 19, 27, 30, 34, 40, 45, 48, 49, 60, 61, 62, 63, 64, 69, 73, 74, 75, 79, 82, 87, 90, 94, 95, 96, 103, 104, 105, 106, 109, 110, 111, 112, 120, 124, 125, 126, 129, 132, 135, 139, 145, 146, 147, 150, 151, 152, 153, 154, 162, 163, 164, 165, 166, 169, 172, 173, 174, 180, 183, 184, 187, 190, 195, 198, 199, … } ;

$\mathfrak{S}_3$ = {49, 61, 62, 63, 64, 74, 75, 95, 96, 104, 105, 106, 110, 111, 112, 125, 126, 146, 147, 151, 152, 153, 154, 163, 164, 165, 166, 173, 174, 184, 199, … } ;

$\mathfrak{S}_4$ = {49, 61, 63, 64, 106, 110, 112, 151, 153, 154, 163, 165, 166, 173, 184, 199, … } ;

$\mathfrak{S}_5$ = {49, 64, 110, 151, 154, 166, 173, 184, 199, … } ;

$\mathfrak{S}_6$ = {49, 64, 110, 151, 154, 166, 173, 184, 199, … } ; $\mathfrak{S}_7$ = {110, 154, 173, … } ;

$\mathfrak{S}_8$ = {154, 173, … } ; $\mathfrak{S}_9$ = {154, … } ; etc.

Since the sets $\mathfrak{S}_k$ exist, they must have a minimal element, which we denote as R_k . Moreover, since $2R_k > p_k$, the sequence $\{R_k\}$ is unbounded and also non-decreasing ; we may expect to see consecutive values of k for which R_k retains the same value. The first 282 values of R_k are given in Table 2 of the Appendix, along with the corresponding values of p_k . The values of R_k for $1 \le k \le 282$ have been submitted to Neil Sloane's web site (q.v. [5]) and are listed there as sequence A051169 ; their monotonic subsequence is also shown there as sequence A051610.

3. <u>ADDITIONAL DEVELOPMENT INVOLVING R</u> :

We introduce a useful definition and lemma.

DEFINITION : An element N of $\mathfrak{S}_k$ is said to be *normal* iff $\theta(2N) \ge 1$;

N is said to be *abnormal* iff $\theta(2N) = 0$.

<u>LEMMA 1</u> : Let $N \in \mathfrak{S}_k$ for some $k \ge 1$, with N normal . Then $N \ge p_{k+1}$.

Proof : Since $\theta(2N) \ge 1$, there exist positive integers i and j such that $2N = p_i + p_j$. However, by the definition of N (as an element of $\mathfrak{S}_k$), we must have $i \ge k+1$ (and $j \ge k+1$, by symmetry) . Hence, $2N \ge 2p_{k+1}$, or $N \ge p_{k+1}$. □

Next, recall the definition of our hypothetical R as the smallest counterexample to SGC (assuming PH) . We let $M = \pi^*(2R)$; note that this implies $p_M < 2R < p_{M+1} < 2p_M$ (the last inequality following from "Bertrand's Postulate"), hence $R < p_M$. We also note that $\delta(2R - p_i) = 0$, $1 \le i \le M$, and $2R > p_M$; therefore, $R \in \mathfrak{S}_M$. By the definition of R_M as the smallest element of $\mathfrak{S}_M$, this implies that $R_M \le R$.

Now if we were to suppose that $R_M < R$, this would imply that R_M is normal (since R is assumed to be the smallest counterexample to SGC), hence LEMMA 1 would imply $R_M \ge p_{M+1}$. This, in turn, would imply the following :

$p_{M+1} \le R_M < R < p_M$, which is impossible. We conclude that PH implies $R = R_M$. We may summarize these results as a second lemma .

<u>LEMMA 2</u> : The smallest counterexample R to SGC , under PH, is also the smallest element of $\mathfrak{S}_M$, namely $R = R_M$, where $M = \pi^*(2R)$, i.e., $p_M < 2R_M < p_{M+1}$. Moreover, under PH, we must have $R_M < p_M$. □

In the next section, we indicate additional development involving the counting functions of the $\mathfrak{S}_k$'s .

4. <u>ADDITIONAL DEVELOPMENT INVOLVING COUNTING FUNCTIONS</u> :

Given a positive integer N, let $A_k(N)$ denote the number of elements of $\mathfrak{S}_k$ that are $\le N$. Clearly, $A_k(N) = 0$ if $1 \le N < R_k$, while $A_k(R_k) = 1$, $k = 1, 2, \ldots$. We also define $\psi_k(n)$ to be the characteristic function of $\mathfrak{S}_k$; that is, $\psi_k(n) = 1$ if $n \in \mathfrak{S}_k$, $\psi_k(n) = 0$ if $n \notin \mathfrak{S}_k$.

It follows that :

$$A_k(N) = \sum_{n = R_k}^{N} \psi_k(n) \qquad (2)$$

It is not difficult to see that we may express the $\psi_k(n)$'s explicitly as follows :

$$\psi_k(n) = \prod_{i=1}^{k} \{1 - \delta(2n - p_i)\} \ , \ n \geq R_k \ , \ k = 1, 2, \ldots \qquad (3)$$

Some important properties of the quantities $A_k(N)$ are noted, valid for all $k \geq 1$:

(i) $A_k(N)$ is a non-negative integer ;

(ii) $A_k(M) \leq A_k(N)$ whenever $M \leq N$;

(iii) $A_k(N) \leq A_{k-1}(N) \leq \ldots \leq A_2(N) \leq A_1(N)$

{Property (iii) follows from the observation that for $k \geq 2$,
$\psi_k(n) = \{1 - \delta(2n - p_k\} \psi_{k-1}(n) \leq \psi_{k-1}(n)$, and also $R_k \geq R_{k-1}\}$;

(iv) $A_k(R_k) = 1$; $A_k(N) = 0$ if $N < R_k$.

Our next goal is to obtain a numerical estimate of the function $A_k(N)$, in terms of a continuous function of N, given the integer k . Not all of the estimates obtained in this section are entirely rigorous, but rather are developed with an aim toward a more rigorous approach, which will be attained in the sequel. Toward this end, we begin by noting that

$$A_1(N) = \sum_{n=6}^{N} \{1 - \delta(2n - 3)\} = N - 5 - \pi(2N - 3) + \pi(7) = N - 1 - \pi(2N - 2) = u(N - 1), \text{ say,}$$

where $u(n) \equiv n - \pi(2n)$, $n \geq 1$. Note that $u(1) = 0$. Also, for all $n \geq 2$, $u(n) - u(n - 1)$ $= 1 - \delta(2n - 1) \geq 0$. Therefore, for n natural, u(n) is non-negative and non-decreasing . By a result [4] due to Rosser and Schoenfeld, $\pi(2n) > 2n/\log(2n)$ if $n \geq 6$. Therefore, for all $N \geq 6$, we have : $A_1(N) = u(N - 1) \leq u(N) < N - 2N/\log(2N)$, or :

$$A_1(N) < N\{1 - 2/\log(2N)\} , N \geq 6 \quad (4)$$

More generally, given an integer k ≥ 1 and any sufficiently large N, we note that

$$A_{k+1}(N) = \sum_{n=R_{k+1}}^{N}\{1-\delta(2n-p_{k+1}\}\Psi_k(n) = \{ \sum_{n=R_{k+1}}^{N}\{1-\delta(2n-p_{k+1}\}\Psi_k(n)\} / \sum_{n=R_{k+1}}^{N}\Psi_k(n) * \{ \sum_{n=R_{k+1}}^{N}\Psi_k(n)\}$$

$$\leq \{ \sum_{n=R_{k+1}}^{N}\{1-\delta(2n-p_{k+1}\}\Psi_k(n)\} / \sum_{N=R_{k+1}}^{N}\Psi_k(n) * \sum_{n=R_k}^{N}\Psi_k(n) = \hat{\Theta}_{k+1}(N) * A_k(N)$$, **where**

$$\hat{\Theta}_{k+1}(N) = \{ \sum_{n=R_{k+1}}^{N}\{1-\delta(2n-p_{k+1}\}\Psi_k(n)\} / \sum_{N=R_{k+1}}^{N}\Psi_k(n) \quad (5)$$

Thus far, we have shown that for any sufficiently large integer N ≥ R_k,

the following is true :

$$A_{k+1}(N) \leq \hat{\Theta}_{k+1}(N)\, A_k(N) \quad (6)$$

Next, define $\Theta_{k+1}(N)$ as follows :

$$\Theta_{k+1}(N) = \{ \sum_{n=R_{k+1}}^{N}\{1-\delta(2n-p_{k+1}\}\} / \sum_{N=R_{k+1}}^{N}1 \quad (7)$$

We note from the definitions in (5) and (7) that $\hat{\Theta}_{k+1}(N)$ and $\Theta_{k+1}(N)$ are "weighted averages" of the quantity 1 - δ(2n – p_{k+1}) , as n varies over the interval [R_{k+1}, N] ; the "weights" are $\Psi_k(n)$ and 1, respectively . Since $\Psi_k(n)$ = 0 or 1, we see the following :

$$\hat{\Theta}_{k+1}(N) \leq \Theta_{k+1}(N) \quad (8)$$

Taking (6) and (8) into account, we may then say that

$$A_{k+1}(N) \leq \Theta_{k+1}(N)\, A_k(N) \quad (9)$$

Now $\Theta_{k+1}(N)$ = {N - R_{k+1} + 1 - π*(2N- p_{k+1}) +π*(2R_{k+1} – 2 - p_{k+1})}/(N - R_{k+1} + 1)

= {N - R_{k+1} + 1 - π(2N- p_{k+1}) +π(2R_{k+1} – 2 - p_{k+1})}/(N - R_{k+1} + 1) .

Since π(2N- p_{k+1}) - π(2R_{k+1} – 2 - p_{k+1}) represents the number of primes in the interval [R_{k+1} - p_{k+1}, N - p_{k+1}] , the "thinning out of primes" property ensures that

this difference is ≥ $\pi(2N) - \pi(2R_{k+1} - 2)$.

Then $\Theta_{k+1}(N) \leq \{N - R_{k+1} + 1 - \pi(2N) + \pi(2R_{k+1} - 2)\}/(N - R_{k+1} + 1)$

$= \{u(N) - u(R_{k+1} - 1)\}/(N - R_{k+1} + 1)$. For sufficiently large N, we may ignore the terms $u(R_{k+1} - 1)$ and $R_{k+1} - 1$ in the last expression, from which it follows that

$$\Theta_{k+1}(N) \leq u(N)/N = \{N - \pi(2N))\}/N \quad (10)$$

We once again cite the result of Rosser and Schoenfeld [4] , that states the following :

$$\pi(N) > N/\log N, \text{ for all } N \geq 11 \quad (11)$$

Combining the results of (9)-(11), we conclude the following :

$$A_{k+1}(N) < (1 - 2/\log 2N) * A_k(N) \quad (12)$$

By an easy inductive process, we obtain the following from (4) and (12), at least for sufficiently large N :

$$A_k(N) < N(1 - 2/\log 2N)^k \quad (13)$$

In TABLE 1 of the APPENDIX, we have indicated the values of $A_k(N)$ for $k = 1, 2, 3, 4$, and $N = 1, 2, \ldots, 200$; also, we have shown the values of $f_k(N)$, where

$$f_k(x) = x\,(1 - 2/\log(2x))^k \quad (14)$$

Although we may compute the values of $f_k(N)$ for $1 \leq N < R_k$, such computation is not very useful, since $A_k(N) = 0$ within such interval ; for this reason, the entries in TABLE 1 are marked "N/A" for the values of $f_k(N)$ in such interval .

Within the limits of TABLE 1, we find that $A_k(N) < f_k(N)$ for *all* N, even when $N = R_k$. However, this relation is illusory ; for numerous values of $k \geq 4$ (*not* indicated in TABLE 1), we find that $f_k(R_k) < 1$.

Indeed, since $\pi(2N) \geq \pi(CN)$ whenever $0 < C < 2$, this leads to

$1 - \pi(2N)/N \leq 1 - \pi(CN)/N$, hence (from Rosser and Schoenfeld's result [4]),

$1 - 2/\log(2N) < 1 - C/\log(CN)$.

5. ADDITIONAL ANALYSIS :

Such considerations suggest that a sequence of positive numbers $\{C_k\}$ exists, such that

$$1/R_k < C_k < \log (C_k R_k) \tag{15}$$

and

$$1 = A_k(R_k) = R_k \{1 - C_k/\log(C_k R_k)\}^k \tag{16}$$

That is, if k and R_k are given, we may determine C_k from the equation given in (16) . Unfortunately, such C_k is *not* uniquely determined, as we will show below. However, we may circumvent this problem by specifying C_k to be the *maximum value* that satisfies (16) ; then, as we will show, C_k is indeed uniquely determined.

In this section, we write C for C_k and R for R_k, for brevity, under the assumption that k and R_k are fixed . Also, we introduce a useful lemma.

<u>LEMMA 3</u> : For all $k \geq 1$ and R (= R_k), we must have $1 - R^{-1/k} > 1/\log R$.
Proof : We note that $1 - R^{-1/k} \geq 1 - R^{-1} = (R - 1)/R$. It suffices to prove that $\log R > \frac{R}{R-1}$. We note that the graphs of the functions log R and $\frac{R}{R-1}$ intersect at a certain unique point ρ, which (as we may verify) is approximately equal to 3.857334827. For all $R > \rho$, we have $\log R > \frac{R}{R-1}$, hence $1 - R^{-1/k} > 1/\log R$. Since the minimum value of R is $R_1 = 6 > \rho$, the Lemma is proved. □

Returning to the relation in (16), we may write it as : $R^k (1 - \frac{C}{\log CR})^k = R^{k-1}$, or $(R - \frac{CR}{\log CR})^k = R^{k-1}$. Hence, $R - \frac{CR}{\log CR} = R^{1-1/k}$, or $\frac{CR}{\log CR} = R(1 - R^{-1/k})$. Using Lemma 3, then $\frac{CR}{\log CR} > \frac{R}{\log R}$. Now consider the function $h(x) = \frac{x}{\log x}$, defined for all $x > 1$. The graph of h is a concave-upward curve, with a minimum value of e at x = e . For all other given ordinates, there are precisely two values of x on the curve , one with $1 < x < e$, the other with $x > e$. For this reason, CR is not uniquely determined by the relation in (16). However, as stated earlier, we may specify C to be the larger of the two solutions implied by (16), in which case, C is uniquely determined. That is, we impose the additional condition $CR > e$. The conditions on C, together with the inequality $\frac{CR}{\log CR} > \frac{R}{\log R}$ imply that $C > 1$.

In TABLE 2, we have indicated the values of R_k , p_k and $C_k > 1$, for $1 \le k \le 282$, which is as far as the available values of R_k have been furnished.

Our ultimate aim is to establish that $R_k > p_k$ for all $k \ge 1$. This result will contradict the property already deduced (see LEMMA 2) as a consequence of PH , namely : $R_M < p_M$. This contradiction will negate PH and thereby establish SGC .

6. PROOF OF SGC :

As we have commented, equation (16) uniquely determines the sequence $\{C_k\}$, with k and R_k given, provided we define such unique value as the larger of the two positive solutions implied by (16) ; indeed, (16) may be taken as the definition of C_k , with the proviso that $C_k R_k > e$.

By taking logarithms in (16), we obtain : 0 = log R_k + k log{1 - C_k /log(C_k R_k)} ,

or, by Taylor series expansion (also using (15)) : log R_k = k $\sum_{n=1}^{\infty} \frac{\left(\frac{C_k}{\log(C_k R_k)}\right)^n}{n}$ **. Thus,**

log R_k > $\frac{k\,C_k}{\log(C_k R_k)}$ **>** $\frac{k}{\log R_k}$ **, since C_k > 1 for all k ≥ 1 . Then $\log^2 R_k$ > k, or**

$$R_k > \exp(\sqrt{k}) , k = 1, 2, \ldots \qquad (17)$$

Now it is readily shown (by graphing or otherwise) that $p_k < \exp(\sqrt{k})$ for all k ≥ 17 ; (for example, it is known that p_k is asymptotic to k log k, which is much less than exp($\sqrt{k}$)) . Also, we may verify from Table 2 that $R_k > p_k$ for 1 ≤ k ≤ 16 . From (17), it then follows that

$$R_k > p_k \text{ for all } k \geq 1 \qquad (18)$$

Clearly, this conclusion contradicts LEMMA 2 (for k = M) ; since LEMMA 2 depends on PH being true, it follows that PH is false. Therefore, SGC is established!

THEOREM : The strong version of the Goldbach Conjecture is true.

7. <u>CONCLUSION</u> :

We have not attempted to verify the so-called "extended" Goldbach Conjecture (EGC) , which is the famous conjecture made by Hardy and Littlewood in [3], giving an asymptotic expression for θ(2n) . This asserts the following :

$$\theta(2n) \sim \frac{2C_2\, n}{(\log n)^2} P(n) , \qquad (19)^*$$

where $C_2 = \prod_{p>2}\left\{1 - \frac{1}{(p-1)^2}\right\}$ is the "Twin Primes" Constant ($C_2 \approx 0.66016$), with the product taken over all odd primes p, and $P(n) = \prod_{p|n, p>2}\left\{\frac{p-1}{p-2}\right\}$, such product taken

over all odd prime divisors of n . The relation in (19)* is an asymptotic one, taken as $n \to \infty$; the asterisk denotes that, as of this writing, it remains an unresolved conjecture. Hardy and Littlewood used heuristic arguments in order to make their various conjectures, including (19)*. It would appear that the derivation of (19)* requires a deeper analysis than is available under the methods indicated in this paper. In any event, EGC does not imply SGC.

It should also be pointed out that SGC implies the "weak" (or "ternary") version of the Goldbach Conjecture (WGC), which asserts that every odd integer ≥ 9 may be written as the sum of three primes. This is actually the original form of the Goldbach Conjecture, first made by Christian Goldbach in a 1742 letter to Euler ; the implication may be verified by taking one of the three primes as 3 .

8 . <u>ACKNOWLEDGMENTS</u> :

The first 282 values of R_k were computed by Dr. Peter G. Anderson, Rochester Institute of Technology, Rochester, New York (with $R_{282} = 923{,}566{,}921$) ; grateful acknowledgment is made to Dr. Anderson for his assistance in this regard. The values of p_k are readily available, while the values of C_k were computed on an Excel worksheet.

Also, grateful acknowledgment is made to Dr. Clark Kimberling of the University of Evansville in Indiana, and to his graduate student J. B. Brown, for computations of the quantities $\Psi_k(n)$ and $A_k(N)$ $(1 \leq n \leq N)$ over an extended range of values of k, n and N, corroborating the assertion made in (13).

We cannot say enough in praise for the inestimable contributions of Dr. John C. Turner from the University of Waikato, New Zealand. Dr. Turner has read various incarnations of this paper and has made a number of cogent

observations and suggestions. Also, Dr. Turner has made invaluable contributions toward the dissemination of this paper's result by placing this on his web site, with considerable investment of his own time and effort.

We also acknowledge the contribution of Dr. Thomas Dence of Ashland University, who has read the manuscript thoroughly and has recommended some significant corrections and additions.

Finally, grateful acknowledgment is made to the anonymous referee, who has recommended certain valuable improvements in the presentation .

APPENDIX

TABLE 1

N	$A_1(N)$	$f_1(N)$	$A_2(N)$	$f_2(N)$	$A_3(N)$	$f_3(N)$	$A_4(N)$	$f_4(N)$
1	0	N/A	0	N/A	0	N/A	0	N/A
2	0	N/A	0	N/A	0	N/A	0	N/A
3	0	N/A	0	N/A	0	N/A	0	N/A
4	0	N/A	0	N/A	0	N/A	0	N/A
5	0	N/A	0	N/A	0	N/A	0	N/A
6	1	1.1708	0	N/A	0	N/A	0	N/A
7	1	1.6951	0	N/A	0	N/A	0	N/A
8	1	2.2292	0	N/A	0	N/A	0	N/A
9	2	2.7724	0	N/A	0	N/A	0	N/A
10	2	3.3238	0	N/A	0	N/A	0	N/A
11	2	3.8827	0	N/A	0	N/A	0	N/A
12	3	4.4482	0	N/A	0	N/A	0	N/A
13	3	5.0199	0	N/A	0	N/A	0	N/A
14	4	5.5972	0	N/A	0	N/A	0	N/A
15	5	6.1796	1	2.5458	0	N/A	0	N/A
16	5	6.7668	1	2.8618	0	N/A	0	N/A
17	5	7.3583	1	3.1850	0	N/A	0	N/A
18	6	7.9540	1	3.5148	0	N/A	0	N/A
19	7	8.5535	2	3.8507	0	N/A	0	N/A
20	7	9.1566	2	4.1922	0	N/A	0	N/A
21	8	9.7631	2	4.5389	0	N/A	0	N/A
22	8	10.3727	2	4.8906	0	N/A	0	N/A
23	8	10.9853	2	5.2468	0	N/A	0	N/A
24	9	11.6007	2	5.6074	0	N/A	0	N/A
25	9	12.2189	2	5.9721	0	N/A	0	N/A
26	10	12.8396	2	6.3406	0	N/A	0	N/A
27	11	13.4627	3	6.7128	0	N/A	0	N/A
28	11	14.0882	3	7.0884	0	N/A	0	N/A
29	12	14.7158	3	7.4675	0	N/A	0	N/A
30	13	15.3456	4	7.8496	0	N/A	0	N/A
31	13	15.9775	4	8.2348	0	N/A	0	N/A
32	13	16.6113	4	8.6229	0	N/A	0	N/A
33	14	17.2469	4	9.0138	0	N/A	0	N/A
34	15	17.8844	5	9.4074	0	N/A	0	N/A
35	15	18.5236	5	9.8035	0	N/A	0	N/A
36	16	19.1645	5	10.2021	0	N/A	0	N/A
37	16	19.8070	5	10.6031	0	N/A	0	N/A
38	16	20.4510	5	11.0064	0	N/A	0	N/A
39	17	21.0966	5	11.4119	0	N/A	0	N/A
40	18	21.7436	6	11.8196	0	N/A	0	N/A

TABLE 1 (Continued)

N	$A_1(N)$	$f_1(N)$	$A_2(N)$	$f_2(N)$	$A_3(N)$	$f_3(N)$	$A_4(N)$	$f_4(N)$
41	18	22.3921	6	12.2294	0	N/A	0	N/A
42	19	23.0419	6	12.6411	0	N/A	0	N/A
43	19	23.6930	6	13.0549	0	N/A	0	N/A
44	20	24.3455	6	13.4705	0	N/A	0	N/A
45	21	24.9992	7	13.8879	0	N/A	0	N/A
46	21	25.6541	7	14.3072	0	N/A	0	N/A
47	22	26.3102	7	14.7282	0	N/A	0	N/A
48	23	26.9674	8	15.1509	0	N/A	0	N/A
49	24	27.6258	9	15.5752	1	8.7812	1	4.9508
50	24	28.2853	9	16.0011	1	9.0519	1	5.1207
51	25	28.9458	9	16.4286	1	9.3243	1	5.2922
52	25	29.6074	9	16.8577	1	9.5983	1	5.4650
53	25	30.2700	9	17.2882	1	9.8738	1	5.6393
54	26	30.9336	9	17.7201	1	10.1509	1	5.8149
55	26	31.5981	9	18.1535	1	10.4294	1	5.9918
56	26	32.2636	9	18.5883	1	10.7094	1	6.1701
57	27	32.9301	9	19.0244	1	10.9908	1	6.3496
58	27	33.5974	9	19.4618	1	11.2736	1	6.5304
59	28	34.2656	9	19.9005	1	11.5577	1	6.7124
60	29	34.9347	10	20.3405	1	11.8432	1	6.8956
61	30	35.6046	11	20.7818	2	12.1299	2	7.0800
62	31	36.2754	12	21.2242	3	12.4180	2	7.2656
63	32	36.9469	13	21.6679	4	12.7073	3	7.4523
64	33	37.6193	14	22.1127	5	12.9979	4	7.6402
65	33	38.2924	14	22.5586	5	13.2896	4	7.8291
66	34	38.9663	14	23.0057	5	13.5825	4	8.0191
67	34	39.6410	14	23.4539	5	13.8766	4	8.2102
68	35	40.3164	14	23.9031	5	14.1719	4	8.4023
69	36	40.9925	15	24.3534	5	14.4682	4	8.5955
70	36	41.6693	15	24.8048	5	14.7657	4	8.7897
71	36	42.3469	15	25.2571	5	15.0642	4	8.9848
72	37	43.0251	15	25.7105	5	15.3638	4	9.1810
73	38	43.7039	16	26.1649	5	15.6645	4	9.3781
74	39	44.3835	17	26.6202	6	15.9662	4	9.5761
75	40	45.0637	18	27.0765	7	16.2689	4	9.7751
76	40	45.7445	18	27.5337	7	16.5726	4	9.9750
77	40	46.4260	18	27.9918	7	16.8772	4	10.1759
78	41	47.1080	18	28.4509	7	17.1829	4	10.3776
79	42	47.7907	19	28.9108	7	17.4895	4	10.5802
80	42	48.4740	19	29.3716	7	17.7970	4	10.7836

TABLE 1 (Continued)

N	$A_1(N)$	$f_1(N)$	$A_2(N)$	$f_2(N)$	$A_3(N)$	$f_3(N)$	$A_4(N)$	$f_4(N)$
81	43	49.1579	19	29.8333	7	18.1054	4	10.9879
82	44	49.8423	19	30.2958	7	18.4148	4	11.1931
83	44	50.5273	20	30.7591	8	18.7250	4	11.3991
84	45	51.2129	21	31.2233	9	19.0362	4	11.6059
85	45	51.8990	21	31.6883	9	19.3481	4	11.8135
86	46	52.5857	21	32.1541	9	19.6610	4	12.0219
87	47	53.2729	21	32.6207	9	19.9747	4	12.2312
88	47	53.9606	21	33.0881	9	20.2892	4	12.4411
89	48	54.6489	21	33.5562	9	20.6046	4	12.6519
90	49	55.3377	21	34.0251	9	20.9208	4	12.8634
91	49	56.0269	21	34.4947	9	21.2377	4	13.0757
92	49	56.7167	21	34.9651	9	21.5555	4	13.2887
93	50	57.4070	21	35.4362	9	21.8740	4	13.5024
94	51	58.0978	21	35.9080	9	22.1933	4	13.7168
95	52	58.7890	21	36.3805	9	22.5134	4	13.9320
96	53	59.4807	21	36.8537	9	22.8342	4	14.1479
97	53	60.1729	21	37.3276	9	23.1558	4	14.3644
98	53	60.8655	21	37.8022	9	23.4781	4	14.5817
99	54	61.5586	21	38.2774	9	23.8011	4	14.7996
100	54	62.2522	21	38.7533	9	24.1248	4	15.0182
101	54	62.9462	21	39.2299	9	24.4492	4	15.2375
102	55	63.6406	21	39.7071	9	24.7743	4	15.4574
103	56	64.3354	22	40.1849	9	25.1002	4	15.6780
104	57	65.0307	23	40.6634	10	25.4266	4	15.8992
105	58	65.7264	24	41.1425	11	25.7538	4	16.1210
106	59	66.4226	25	41.6222	12	26.0816	5	16.3435
107	59	67.1191	25	42.1025	12	26.4101	5	16.5666
108	60	67.8160	25	42.5835	12	26.7393	5	16.7903
109	61	68.5134	26	43.0650	12	27.0691	5	17.0146
110	62	69.2111	27	43.5471	13	27.3995	6	17.2395
111	63	69.9093	28	44.0298	14	27.7305	6	17.4651
112	64	70.6078	29	44.5130	15	28.0622	7	17.6912
113	64	71.3067	29	44.9969	15	28.3945	7	17.9179
114	65	72.0060	29	45.4812	15	28.7274	7	18.1451
115	65	72.7057	29	45.9662	15	29.0609	7	18.3730
116	65	73.4057	29	46.4517	15	29.3950	7	18.6014
117	66	74.1061	29	46.9377	15	29.7297	7	18.8303
118	66	74.8069	29	47.4243	15	30.0649	7	19.0599
119	67	75.5080	29	47.9114	15	30.4008	7	19.2899
120	68	76.2095	30	48.3990	15	30.7372	7	19.5206

TABLE 1 (Continued)

N	$A_1(N)$	$f_1(N)$	$A_2(N)$	$f_2(N)$	$A_3(N)$	$f_3(N)$	$A_4(N)$	$f_4(N)$
121	68	76.9113	30	48.8872	15	31.0742	7	19.7517
122	68	77.6135	30	49.3759	15	31.4118	7	19.9834
123	69	78.3160	30	49.8651	15	31.7499	7	20.2156
124	70	79.0189	31	50.3548	15	32.0885	7	20.4484
125	71	79.7221	32	50.8449	16	32.4277	7	20.6817
126	72	80.4257	33	51.3356	17	32.7675	7	20.9155
127	72	81.1296	33	51.8268	17	33.1078	7	21.1497
128	73	81.8338	33	52.3185	17	33.4486	7	21.3845
129	74	82.5383	34	52.8106	17	33.7899	7	21.6199
130	74	83.2431	34	53.3032	17	34.1318	7	21.8557
131	75	83.9483	34	53.7963	17	34.4741	7	22.0919
132	76	84.6538	35	54.2899	17	34.8170	7	22.3287
133	76	85.3596	35	54.7839	17	35.1604	7	22.5660
134	77	86.0657	35	55.2784	17	35.5043	7	22.8037
135	78	86.7721	36	55.7733	17	35.8487	7	23.0420
136	78	87.4788	36	56.2687	17	36.1936	7	23.2807
137	78	88.1859	36	56.7646	17	36.5389	7	23.5198
138	79	88.8932	36	57.2609	17	36.8848	7	23.7595
139	80	89.6008	37	57.7576	17	37.2311	7	23.9995
140	80	90.3087	37	58.2547	17	37.5779	7	24.2401
141	81	91.0169	37	58.7523	17	37.9252	7	24.4811
142	81	91.7254	37	59.2503	17	38.2730	7	24.7225
143	81	92.4342	37	59.7488	17	38.6212	7	24.9644
144	82	93.1432	37	60.2476	17	38.9698	7	25.2068
145	83	93.8525	38	60.7469	17	39.3190	7	25.4496
146	84	94.5622	39	61.2466	18	39.6686	7	25.6928
147	85	95.2720	40	61.7467	19	40.0186	7	25.9364
148	85	95.9822	40	62.2472	19	40.3691	7	26.1805
149	86	96.6926	40	62.7481	19	40.7200	7	26.4250
150	87	97.4033	41	63.2494	19	41.0713	7	26.6699
151	88	98.1143	42	63.7511	20	41.4231	8	26.9152
152	89	98.8255	43	64.2532	21	41.7753	8	27.1610
153	90	99.5370	44	64.7557	22	42.1280	9	27.4072
154	91	100.2488	45	65.2585	23	42.4811	10	27.6537
155	91	100.9608	45	65.7618	23	42.8346	10	27.9007
156	92	101.6730	45	66.2654	23	43.1885	10	28.1481
157	92	102.3855	45	66.7694	23	43.5428	10	28.3959
158	92	103.0983	45	67.2738	23	43.8976	10	28.6441
159	93	103.8113	45	67.7786	23	44.2527	10	28.8927
160	93	104.5246	45	68.2837	23	44.6083	10	29.1416

TABLE 1 (Continued)

N	$A_1(N)$	$f_1(N)$	$A_2(N)$	$f_2(N)$	$A_3(N)$	$f_3(N)$	$A_4(N)$	$f_4(N)$
161	94	105.2381	45	68.7892	23	44.9642	10	29.3910
162	95	105.9518	46	69.2950	23	45.3206	10	29.6407
163	96	106.6658	47	69.8012	24	45.6773	11	29.8909
164	97	107.3801	48	70.3078	25	46.0345	11	30.1414
165	98	108.0945	49	70.8147	26	46.3920	12	30.3923
166	99	108.8093	50	71.3220	27	46.7500	13	30.6435
167	99	109.5242	50	71.8296	27	47.1083	13	30.8952
168	100	110.2394	50	72.3376	27	47.4670	13	31.1472
169	101	110.9548	51	72.8459	27	47.8261	13	31.3996
170	101	111.6704	51	73.3546	27	48.1855	13	31.6523
171	102	112.3863	51	73.8636	27	48.5453	13	31.9054
172	103	113.1024	52	74.3729	27	48.9055	13	32.1589
173	104	113.8187	53	74.8826	28	49.2661	14	32.4127
174	105	114.5352	54	75.3926	29	49.6271	14	32.6669
175	105	115.2520	54	75.9029	29	49.9884	14	32.9215
176	105	115.9689	54	76.4136	29	50.3500	14	33.1764
177	106	116.6861	54	76.9246	29	50.7121	14	33.4316
178	106	117.4035	54	77.4359	29	51.0744	14	33.6872
179	107	118.1212	54	77.9475	29	51.4372	14	33.9431
180	108	118.8390	55	78.4595	29	51.8003	14	34.1994
181	108	119.5571	55	78.9718	29	52.1637	14	34.4560
182	109	120.2753	55	79.4843	29	52.5275	14	34.7130
183	110	120.9938	56	79.9972	29	52.8916	14	34.9703
184	111	121.7125	57	80.5104	30	53.2561	15	35.2279
185	111	122.4313	57	81.0240	30	53.6209	15	35.4858
186	112	123.1504	57	81.5378	30	53.9861	15	35.7441
187	113	123.8697	58	82.0519	30	54.3516	15	36.0027
188	113	124.5892	58	82.5663	30	54.7174	15	36.2617
189	114	125.3089	58	83.0810	30	55.0836	15	36.5210
190	115	126.0288	59	83.5961	30	55.4501	15	36.7805
191	115	126.7489	59	84.1114	30	55.8169	15	37.0405
192	116	127.4692	59	84.6270	30	56.1840	15	37.3007
193	116	128.1896	59	85.1429	30	56.5515	15	37.5612
194	117	128.9103	59	85.6591	30	56.9193	15	37.8221
195	118	129.6312	60	86.1756	30	57.2874	15	38.0832
196	118	130.3522	60	86.6924	30	57.6558	15	38.3447
197	119	131.0735	60	87.2094	30	58.0246	15	38.6065
198	120	131.7949	61	87.7268	30	58.3937	15	38.8686
199	121	132.5165	62	88.2444	31	58.7630	16	39.1310
200	121	133.2384	62	88.7623	31	59.1327	16	39.3937

TABLE 2

k	R_k	p_k	C_k	k	R_k	p_k	C_k
1	6	3	2.118868	41	21766	181	2.343662
2	15	5	2.762644	42	21766	191	2.289268
3	49	7	3.798092	43	21766	193	2.237211
4	49	11	3.130766	44	21766	197	2.187345
5	49	13	2.627296	45	21766	199	2.139537
6	49	17	2.242826	46	27122	211	2.187352
7	110	19	2.802842	47	27122	223	2.141556
8	154	23	2.841151	48	27122	227	2.097541
9	154	29	2.562054	49	27122	229	2.055207
10	278	31	2.876681	50	31637	233	2.076290
11	278	37	2.642867	51	31637	239	2.035971
12	278	41	2.440752	52	31637	241	1.997110
13	278	43	2.264589	53	31637	251	1.959629
14	496	47	2.559082	54	31637	257	1.923460
15	496	53	2.399695	55	31637	263	1.888534
16	496	59	2.257371	56	31637	269	1.854790
17	496	61	2.129608	57	31637	271	1.822170
18	496	67	2.014355	58	31637	277	1.790620
19	496	71	1.909920	59	31637	281	1.760089
20	1321	73	2.438069	60	31637	283	1.730529
21	1321	79	2.327273	61	56836	293	1.904477
22	1321	83	2.225390	62	56836	307	1.873761
23	1321	89	2.131416	63	56836	311	1.843973
24	1321	97	2.044491	64	64084	313	1.855894
25	1321	101	1.963872	65	64084	317	1.827271
26	2686	103	2.284317	66	97214	331	1.940081
27	2686	107	2.202221	67	97214	337	1.911172
28	2686	109	2.125435	68	97214	347	1.883072
29	2686	113	2.053474	69	97214	349	1.855749
30	2686	127	1.985908	70	97214	353	1.829172
31	2686	131	1.922355	71	97235	359	1.803379
32	2686	137	1.862478	72	97235	367	1.778205
33	3713	139	1.960720	73	97235	373	1.753692
34	3713	149	1.903375	74	97235	379	1.729815
35	3713	151	1.849078	75	206786	383	1.942931
36	3713	157	1.797597	76	251611	389	1.980171
37	3713	163	1.748724	77	251611	397	1.954532
38	3713	167	1.702270	78	251611	401	1.929521
39	21766	173	2.460098	79	251611	409	1.905115
40	21766	179	2.400548	80	251611	419	1.881293

TABLE 2 (Continued)

k	R_k	p_k	C_k	k	R_k	p_k	C_k
81	251611	421	1.858035	121	1763479	673	1.669359
82	251611	431	1.835320	122	1763479	677	1.655542
83	251611	433	1.813131	123	1763479	683	1.641944
84	251611	439	1.791450	124	1763479	691	1.628559
85	251611	443	1.770259	125	1763479	701	1.615383
86	251611	449	1.749542	126	1763479	709	1.602411
87	251611	457	1.729285	127	1763479	719	1.589638
88	251611	461	1.709471	128	1903702	727	1.594319
89	251611	463	1.690088	129	1903702	733	1.581804
90	251611	467	1.671120	130	1903702	739	1.569477
91	251611	479	1.652556	131	1903702	743	1.557334
92	251611	487	1.634382	132	5379961	751	1.780817
93	251611	491	1.616587	133	5379961	757	1.767358
94	251611	499	1.599159	134	5379961	761	1.754094
95	251611	503	1.582087	135	5379961	769	1.741021
96	251611	509	1.565360	136	5379961	773	1.728135
97	251611	521	1.548968	137	5379961	787	1.715432
98	538711	523	1.731865	138	5379961	797	1.702908
99	538711	541	1.714235	139	5379961	809	1.690559
100	538711	547	1.696946	140	5379961	811	1.678382
101	538711	557	1.679988	141	5379961	821	1.666374
102	538711	563	1.663353	142	5379961	823	1.654530
103	538711	569	1.647030	143	5379961	827	1.642848
104	538711	571	1.631013	144	12053441	829	1.808941
105	538711	577	1.615291	145	12053441	839	1.796415
106	538711	587	1.599858	146	12053441	853	1.784056
107	538711	593	1.584705	147	12053441	857	1.771860
108	538711	599	1.569825	148	12053441	859	1.759824
109	1763479	601	1.854295	149	12053441	863	1.747946
110	1763479	607	1.837394	150	12053441	877	1.736222
111	1763479	613	1.820787	151	12053441	881	1.724649
112	1763479	617	1.804467	152	12053441	883	1.713224
113	1763479	619	1.788426	153	12053441	887	1.701945
114	1763479	631	1.772657	154	12053441	907	1.690809
115	1763479	641	1.757154	155	12053441	911	1.679812
116	1763479	643	1.741910	156	12053441	919	1.668954
117	1763479	647	1.726918	157	13894939	929	1.687847
118	1763479	653	1.712173	158	13894939	937	1.677078
119	1763479	659	1.697669	159	13894939	941	1.666441
120	1763479	661	1.683399	160	13894939	947	1.655934

TABLE 2 (Continued)

k	R_k	p_k	C_k	k	R_k	p_k	C_k
161	13894939	953	1.645554	201	93926431	1231	1.645905
162	13894939	967	1.635300	202	93926431	1237	1.637687
163	13894939	971	1.625169	203	93926431	1249	1.629548
164	13894939	977	1.615158	204	93926431	1259	1.621487
165	13894939	983	1.605267	205	93926431	1277	1.613504
166	13894939	991	1.595491	206	93926431	1279	1.605596
167	18999469	997	1.648063	207	93926431	1283	1.597763
168	18999469	1009	1.638163	208	93926431	1289	1.590005
169	18999469	1013	1.628376	209	93926431	1291	1.582319
170	18999469	1019	1.618703	210	93926431	1297	1.574705
171	18999469	1021	1.609140	211	93926431	1301	1.567161
172	18999469	1031	1.599686	212	93926431	1303	1.559688
173	18999469	1033	1.590339	213	93926431	1307	1.552284
174	30059956	1039	1.670691	214	93926431	1319	1.544947
175	30059956	1049	1.661065	215	167535419	1321	1.638173
176	30059956	1051	1.651546	216	167535419	1327	1.630523
177	30059956	1061	1.642132	217	167535419	1361	1.622943
178	30059956	1063	1.632821	218	167535419	1367	1.615431
179	30059956	1069	1.623612	219	167535419	1373	1.607987
180	30059956	1087	1.614503	220	167535419	1381	1.600608
181	30059956	1091	1.605493	221	167535419	1399	1.593296
182	56816411	1093	1.719169	222	167535419	1409	1.586048
183	56816411	1097	1.709712	223	167535419	1423	1.578863
184	56816411	1103	1.700357	224	209955962	1427	1.610172
185	56816411	1109	1.691100	225	209955962	1429	1.602949
186	56816411	1117	1.681940	226	209955962	1433	1.595788
187	56816411	1123	1.672876	227	209955962	1439	1.588690
188	56816411	1129	1.663907	228	209955962	1447	1.581653
189	56816411	1151	1.655030	229	209955962	1451	1.574676
190	56816411	1153	1.646245	230	209955962	1453	1.567759
191	93926431	1163	1.732717	231	209955962	1459	1.560901
192	93926431	1171	1.723638	232	209955962	1471	1.554101
193	93926431	1181	1.714650	233	209955962	1481	1.547358
194	93926431	1187	1.705754	234	209955962	1483	1.540672
195	93926431	1193	1.696947	235	209955962	1487	1.534042
196	93926431	1201	1.688227	236	209955962	1489	1.527467
197	93926431	1213	1.679595	237	209955962	1493	1.520947
198	93926431	1217	1.671047	238	209955962	1499	1.514481
199	93926431	1223	1.662584	239	209955962	1511	1.508068
200	93926431	1229	1.654204	240	209955962	1523	1.501707

TABLE 2 (Continued)

k	R_k	p_k	C_k
241	209955962	1531	1.495399
242	209955962	1543	1.489142
243	209955962	1549	1.482935
244	209955962	1553	1.476779
245	209955962	1559	1.470672
246	209955962	1567	1.464615
247	209955962	1571	1.458605
248	209955962	1579	1.452643
249	360506719	1583	1.531408
250	360506719	1597	1.525213
251	360506719	1601	1.519067
252	360506719	1607	1.512970
253	360506719	1609	1.506919
254	360506719	1613	1.500915
255	360506719	1619	1.494958
256	360506719	1621	1.489047
257	360506719	1627	1.483181
258	360506719	1637	1.477360
259	360506719	1657	1.471583
260	360506719	1663	1.465850
261	360506719	1667	1.460160
262	360506719	1669	1.454513
263	360506719	1693	1.448909
264	360506719	1697	1.443346
265	360506719	1699	1.437825
266	360506719	1709	1.432345
267	360506719	1721	1.426905
268	360506719	1723	1.421505
269	360506719	1733	1.416145
270	360506719	1741	1.410824
271	360506719	1747	1.405542
272	360506719	1753	1.400298
273	360506719	1759	1.395093
274	360506719	1777	1.389924
275	360506719	1783	1.384793
276	360506719	1787	1.379699
277	923566921	1789	1.512274
278	923566921	1801	1.506772
279	923566921	1811	1.501310
280	923566921	1823	1.495886
281	923566921	1831	1.490500
282	923566921	1847	1.485152

REFERENCES

1. **P. S. Bruckman. "A New Insight into the Goldbach Conjecture". Pi Mu Epsilon Journal, 10.10 (1999), 791-795**

2. **P.S. Bruckman. "A Proof of the Strong Goldbach Conjecture".** *The International Journal of Mathematical Education in Science and Technology.* **Published as a Class Note 5/09/08, in iFirst, pp. 1-8. Later published in the regular journal, Vol. 39.8 (October 2008) : 1102-1109.**

3. **G. H. Hardy & J. E. Littlewood. "Some problems of 'Partitio Numerorum III : On the expression of a number as a sum of primes'. Acta Math. (1923) 44, 1-70. Reprinted in "Collected Works of G. H. Hardy", Vol. I : 561-630, Clarendon Press, Oxford, 1966.**

4. **J. B. Rosser & L. Schoenfeld. "Approximate Formulas for some Functions of Prime Numbers". Ill. J. Math., Vol. 6 (1962), 64-94.**

5. **N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. http://www.research.att.com/~njas/ ; Sequences A051169 and A051610.**

6. **W. Yuan (ed.), Goldbach Conjecture, World Scientific, Singapore, 1964.**

MATHEMATICS SUBJECT CLASSIFICATIONS : 11P32, 11A41