A NEW PROOF OF THE COLLATZ CONJECTURE

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ABSTRACT : This paper is a revised version of a previous paper by the author that, after publication, was found to contain an error. This paper supersedes and replaces the previous paper. A modified form of the Collatz transformation is formulated, leading to the concept of a modified Collatz chain. A smallest counterexample N_0 to the Collatz Conjecture is hypothesized ; the existence of N_0 implies that N_0 generates an infinite sequence $\{N_k\}$, each of whose elements is at least as large as N_0 , as well as certain auxiliary sequences $\{f_k\}$, $\{F_k\}$ and $\{S_k\}$. A formula for N_k is derived, dependent on the starting value N_0. We show that $k \le F_k \le Ck$ for all k, where C = log 3/log 2, and also that N_0 must be unbounded as k increases, which contradicts the requirement that N_0 be fixed, and therefore bounded . This contradiction establishes the Collatz Conjecture.

KEY WORDS OR TERMS : Collatz transformation ; modified Collatz transformation ; Collatz chain ; modified Collatz chain ; auxiliary sequence.

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1. INTRODUCTION :

This paper is a revised version of a paper [1] by the author that was previously published as a proof of the Collatz Conjecture, then subsequently found to contain an error that negated such proof . The error was first brought to the author's attention by Mr. Jeff Norris of Paris Junior College, and later by Dr. Jeffrey C. Lagarias of the University of Michigan in a bibliographical paper [2] , which also contains mention of reference [3]. Unfortunately, the error was not noticed until after publication of the original paper.

The error discovered in the author's original paper [1] has essentially been circumvented and eliminated in this paper, since a somewhat different and more direct method of proof has been used. However, much of the original notation previously adopted, along with the preliminary discussion, has been retained in this paper. Nevertheless, it has been deemed prudent to "start afresh" and redefine the appropriate terms for the purposes of clarity, as if this were a brand-new paper . In effect, it is as if the original paper [1] had never existed.

Much literature has emerged in recent years regarding the Collatz conjecture (CC), also known as the "3x+1" conjecture. A recent paper [3] by Shaw gives a

brief history of the problem and of some of the attempts to solve it . A more complete bibliography of the prior efforts made to prove CC is given in [2].

Simply stated, if we begin with an arbitrary positive integer n, we define the "Collatz transformation" T(n) as follows :

If n is even, $T(n) = n/2$; if n is odd, $T(n) = 3n+1$ (1)

In the sequel, variables denote positive integers, unless otherwise indicated.

The k-th convolution of the Collatz transformation is denoted as $T^k(n)$.

CC states that for all $n \geq 1$, there exists a positive integer k such that $T^k(n) = 1$.

This author is pleased to report that an elementary (and simplified) proof of CC has been found, and this is presented in this paper.

2. THE MODIFIED COLLATZ TRANSFORMATION :

We find it convenient to modify the Collatz transformation in a certain way . The "modified" Collatz transformation U(n) is defined as follows for all *odd* $n \geq 1$:

$$U(n) = (3n+1)/2^e, \text{ where } 2^e \,\|\, (3n+1) \quad (2)$$

That is, e is the highest exponent of 2 dividing 3n+1 . Let $U^k(n)$ denote the k-th convolution of U. We then see that $U^k(n)$ is odd for all odd n and all k. Essentially, U bypasses the steps of division by powers of 2 that are part and parcel of the T-transformation, since these steps are incorporated at once in the U-transformation. We also see that CC is equivalent to the statement that for all odd $n \geq 1$, there exists a positive integer k such that $U^k(n) = 1$.

We note that for all k and all odd n, there exists an exponent $e = e(n,k)$ such that

$$U^{k+1}(n) = \{(3U^k(n)+1\}/2^e \text{ , and } 2^e \,\|\, \{3U^k(n)+1\} \qquad (3)$$

Thus, a "modified Collatz chain" {n, U(n), $U^2(n)$, $U^3(n)$, …} is generated by applying the modified Collatz transformation repeatedly, beginning with a given odd n ; note that each exponent "e" in this relation is ≥ 1. We also see that a given starting value of n generates an"auxiliary chain", or sequence {e(n, 1), e(n, 2), e(n, 3), …} ; it is possible, *a priori*, that the same auxiliary chain might be generated by another starting value of n. CC states that such modified Collatz chains might be considered to terminate with a "1" and therefore to be necessarily finite ; since U(1) = 1, there might seem to be no point in continuing the Collatz chain after the first occurrence of a "1". This, of course, also applies to the corresponding auxiliary chains. Nevertheless, for our purposes, we find it convenient to consider a modified Collatz chain as infinite under *any* circumstances. If $n_0 > 1$ is our starting value, and if some $r \geq 1$ exists such that $U^k(n_0) = 1$ for all $k \geq r$ and $U^k(n_0) > 1$ for all k with $0 \leq k < r$, we call n_0 a "normal" number ; that is, we deem the modified Collatz chain to be infinite, even for normal numbers. For our purposes, we also deem $n_0 = 1$ to be a normal number. For all normal numbers, we note that the corresponding values of e_k are equal to 2 for all $k \geq r$. By contrast, if $U^k(n_0) > 1$ for all $k \geq 0$, we call n_0 an "abnormal" number .

Thus, CC is equivalent to stating that there exists no abnormal n . For brevity, we write $U^k(n_0) = n_k$, with $U^0(n) = n = n_0$, the starting value.

By repeated application of the recurrence given in (3), we obtain the following, for some sequence $\{e_k\}$ and starting value n_0 : $n_1 = (3n_0+1)/2^{e_1}$; $n_2 = (3n_1 + 1)/2^{e_2} = (3^2n_0 + 3 + 2^{e_1})/2^{e_1+e_2}$; $n_3 = (3n_2 + 1)/2^{e_3}$ $= (3^3n_0 + 3^2 + 3*2^{e_1} + 2^{e_1+e_2})/2^{e_1+e_2+e_3}$; etc. For brevity, we write $E_k = e_1+ e_2+ e_3+\ldots+ e_k$. In general, we obtain the following relation satisfied by n_k and n_0 :

$$2^{E_k} n_k - 3^k n_0 = s_k , k = 0, 1, 2, \ldots, \qquad (4)$$

where

$$s_k = \sum_{j=0}^{k-1} 3^{k-1-j} 2^{E_j} \qquad (5)$$

For completeness, we define $e_0 = E_0 = s_0 = 0$. Clearly, given n_0, this determines the "auxiliary" sequences $\{e_k\}$, $\{E_k\}$ and $\{s_k\}$. We may obtain successive values of s_k (given $\{E_k\}$) from the following (easily verified) recurrence relation :

$$s_{k+1} = 3s_k + 2^{E_k} , k = 0, 1, \ldots , \text{with } s_0 = 0 . \qquad (6)$$

Again, the sequences $\{e_k\}$, $\{E_k\}$ and $\{s_k\}$ are determined by n_0. If n_0 is normal, the sequence $\{e_k\}$ contains an infinite string of "2"'s for all sufficiently large k . Our aim, of course, is to show that this latter situation must always prevail.

3. PROPERTIES OF THE SMALLEST ABNORMAL NUMBER :

In this section, we will suppose that $N_0 > 1$ is the smallest abnormal number. Moreover, to distinguish the auxiliary sequences corresponding to N_0 from all other auxiliary sequences, we will denote these as $\{f_k\}$,$\{F_k\}$ and $\{S_k\}$, with the customary meanings and relations assigned as above.

By definition of N_0, we require that $N_k = U^k(N_0) \geq N_0$ for all k , where all N_k's will also be abnormal . Our aim is to show that the initial assumption as to the existence of N_0 leads to a contradiction .

We begin with the relation in (4), which we may express as follows :

$$2^{F_k} N_k - 3^k N_0 = S_k \text{ , } k = 0, 1, \ldots \qquad (7)$$

We also have :

$$S_k = \sum_{j=0}^{k-1} 3^{k-1-j}\, 2^{F_j} \text{ , } k = 1, 2, \ldots, \text{ with } S_0 = 0 . \qquad (8)$$

$$S_{k+1} = 3S_k + 2^{F_k} \text{ , } k = 0, 1, \ldots \qquad (9)$$

We already know that 1 is a normal number. Note that for $n_0 = 1$, we have $E_k = 2k$ for all $k \geq 0$. For this special case, using (5), we find that $s_k = 4^k - 3^k$. Applying the formula in (4), we have (for $n_0 = 1$) : $4^k n_k - 3^k = 4^k - 3^k$, which implies $n_k = 1$ for all $k \geq 0$, as expected.

We now explore the properties that N_0 must satisfy ; also, we seek to determine the conditions that the auxiliary sequences corresponding to N_0 must satisfy.

Suppose initially that N_0 is such that the sequence $\{f_k\}$ is the minimum possible ; that is to say, let us assume that $f_k = 1$ for all $k \geq 1$. Then $F_k = k$ for all $k \geq 0$. Applying the formula in (8), we find that $S_k = 3^k - 2^k$ for all $k \geq 0$. Then, applying the formula in (7), we obtain : $2^k(N_k + 1) = 3^k(N_0 + 1)$. The general "solution" $\{N_0 , N_k\}$ of this last Diophantine equation yields : $N_0 = -1 + p_k 2^k$, $N_k = -1 + p_k 3^k$, where p_k is a positive integer (since N_0 and N_k are to be positive). However, we also require that N_0 and N_k are to be odd, and

that N_0 be the smallest abnormal number. These conditions force us to set p_k to be some even number for all $k \geq 0$. Therefore, under this scenario, we deduce that $N_0 = A_k * 2^{k+1} - 1$ for some positive integer A_k ; this, however, is absurd, since N_0 is assumed to be a fixed integer, and therefore bounded (with increasing k) . We must therefore exclude the possibility that $f_k = 1$ for all $k \geq 1$.

Next, let us suppose that N_0 is such that each f_k is the *maximum* possible. Applying the formulas in (7) and (9), along with the additional property that $N_k \geq N_0$, we obtain : $S_1 = 1$, $N_1 = (3N_0+1)/2^{F_1} \geq N_0$, hence $F_1 = 1$; then $S_2 = 5$, $N_2 = (9N_0+5)/2^{F_2} \geq N_0$, hence $F_2 = 3$; then $S_3 = 23$, $N_3 = (27N_0+23)/2^{F_3} \geq N_0$, hence $F_3 = 4$; etc. In general, the "maximal" sequence $\{F_k\}$ is obtained by requiring the following conditions :

$$2^{F_k} < 3^k < 2^{1+F_k} \qquad \textbf{(10)}$$

We find that F_k is uniquely determined from this last relation ; namely :

$$F_k = [Ck] \text{ , where } C = \log 3/\log 2 \approx 1.58496250 \qquad \textbf{(11)}$$

Note that the values of f_k (for $k \geq 1$) obtained from (11) (for the maximal sequence $\{f_k\}$) are either 1 or 2 . This implies that for our putative smallest abnormal number N_0, the corresponding auxiliary sequence $\{f_k\}$ (for $k \geq 1$) must consist entirely of 1's or 2's, with $f_1 = 1$, and it must contain at least one 2 .

4. PROOF OF THE COLLATZ CONJECTURE :

Next, suppose that N_0 is such that $\{F_k\}$ is either the so-called "maximal" sequence in (11), or an "intermediate" sequence (where at least one, but not all of

the f_k's equal to 2 are changed to a 1). We see from (8) (or (9)) that S_k is completely determined as a function of $\{F_k\}$, and we may regard (7) as a Diophantine equation to be solved for N_0 and N_k, valid for all $k \geq 0$.

We also note that $\gcd\{S_k, 6\} = 1$, which is actually true for all starting values N_0, or from our new outlook, for all starting auxiliary sequences $\{F_k\}$. Since 2^{F_k}, 3^k and S_k are mutually coprime, we see that (7) always has "solutions" $\{N_k, N_0\}$ for any given k . Likewise, it is true that the following equation always has solutions $\{u_k, v_k\}$ if $\{F_k\}$ is given by (11), in which case S_k is given :

$$3^k v_k - 2^{F_k} u_k = S_k \ , \ k = 0, 1, \dots \ ; \qquad (12)$$

For any given $\{F_k\}$, $k \geq 1$, let us denote the minimum positive solutions of (12) as $\{u_k, v_k\}$; note that this requires one or both of the conditions : $0 < u_k < 3^k$, $0 < v_k < 2^{F_k}$. We also define $u_0 = v_0 = 0$. We see from (12) that v_k must necessarily be odd for all $k \geq 1$, since S_k is odd. However, since u_k is the minimum positive solution of the congruence $2^{F_k} u_k \equiv -S_k \pmod{3^k}$, it may be either odd or even .

Replacing k by k+1 in (12), we obtain : $3^{k+1} v_{k+1} - 2^{F_{k+1}} u_{k+1} = S_{k+1}$. Now using the relation in (9) and subtracting, we obtain :

$3^{k+1} v_{k+1} - 2^{F_{k+1}} u_{k+1} = 3\,\{3^k v_k - 2^{F_k} u_k\} + 2^{F_k}$, which yields :

$3^{k+1}\{ v_{k+1} - v_k \} = 2^{F_k}\{2^{f_{k+1}} u_{k+1} - (3u_k - 1)\}$. This implies that $v_{k+1} \equiv v_k \pmod{2^{F_k}}$, and $2^{f_{k+1}} u_{k+1} \equiv 3u_k - 1 \pmod{3^{k+1}}$. More importantly, it is clear by comparison of (7) and (12) that $N_k \equiv -u_k \pmod{3^k}$ and $N_0 \equiv -v_k \pmod{2^{F_k}}$.

There are infinitely many k such that $0 < v_k < 2^{F_k}$ and v_k is odd .

Indeed, inspection of Table 1 in the Appendix shows that $0 < v_k < 2^{F_k}$ for all $0 < k \leq 25$. For all such k, then $N_0 = B_k*2^{1+F_k} - v_k$, where B_k is a positive integer (since N_0 must be odd), hence $N_0 > 2^{F_k}$. However, since N_0 must be bounded, this is clearly absurd.

This eliminates all of the possible cases under the assumption that N_0 exists. Since the existence of a finite value for N_0 implies that N_0 is unbounded, the contradiction establishes CC.

In Table 1 of the Appendix, we indicate the values of v_k and u_k, along with the values of k and S_k, if $F_k = [Ck]$ for k = 0, 1, …, 25 . It is evident that $v_k \equiv v_j \pmod{2^{F_k}}$ for all $0 \leq j \leq k$.

As we have shown, for any starting value n_0, the auxiliary sequence $\{e_k\}$ must consist entirely of "2"'s for all sufficiently large values of k. That is, there are no abnormal numbers.

APPENDIX
Table 1

k	$F_k = [Ck]$	u_k	v_k	S_k
0	0	0	0	0
1	1	1	1	1
2	3	5	5	5
3	4	7	5	23
4	6	5	5	85
5	7	7	5	319
6	9	5	5	1085
7	11	1097	1029	3767
8	12	1645	1029	13349
9	14	11075	9221	44143
10	15	16612	9221	148813
11	17	145319	107525	479207
12	19	108989	107525	1568693
13	20	163483	107525	5230367
14	22	122612	107525	16739677
15	23	7358371	4301829	54413335
16	25	5518778	4301829	171628613
17	26	72848248	37856261	548440271
18	28	151491308	104965125	1712429677
19	30	985314581	910271493	5405724487
20	31	1477971871	910271493	17290915285
21	33	1108478903	910271493	54020229503
22	34	1662718354	910271493	170650623101
23	36	24782833472	18090140677	529131738487
24	38	89194509224	86809617413	1656114692197
25	39	557436068557	361687524357	5243221983535

$C = \log 3/\log 2 \approx 1.58496250$; $F_k = [Ck]$;

$2^{[Ck]}\, u_k - 3^k\, v_k = S_k$, $k \geq 1$; $u_0 = v_0 = 0$;

for $k \geq 1$, u_k and v_k are the minimum positive integers satisfying this equation ;

$v_{25} = 1 + 2^2 + 2^{10} + 2^{13} + 2^{15} + 2^{16} + 2^{22} + 2^{25} + 2^{26} + 2^{28} + 2^{29} + 2^{34} + 2^{36} + 2^{38}$

REFERENCES

[1] P. S. Bruckman. "A Proof of the Collatz Conjecture". *The International Journal for Mathematical Education in Science and Technology*. Vol. 39.3 : 403-407.

[2] J. C. Lagarias. "The 3x+1 Problem : An Annotated Bibliography, II (2001 -)". arXiv:math/0608208v3 [math.NT] , last rev. April 22, 2008 : pp. 1-25.

[3] D. J. Shaw. "The Pure Numbers Generated by the Collatz Sequence". *The Fibonacci Quarterly* 44.3 (2006) : 194-201.

MATHEMATICS SUBJECT CLASSIFICATIONS : 11B37, 11B83